Patterns in Prime Numbers and the Riemann Hypothesis

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Warwick Maths Society

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What is number theory?

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Number theory is the study of whole numbers

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and the various patterns that exist in the world of numbers.

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6, 28, 496, 8128, ...	(Perfect numbers)
2, 3, 5, 7, 11, 13, 17, 19, ...	(Prime numbers)

Patterns in number sequences

$$1 = 1$$

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$$1 + 3 = 4$$

Patterns in number sequences

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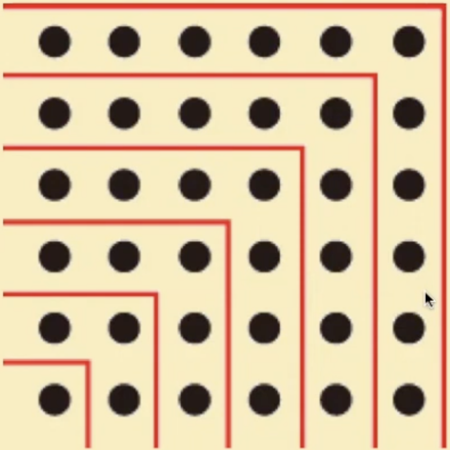
$$\cdots$$

$$1 + 3 + \cdots + (2n - 1) = n^2$$

A pictorial proof



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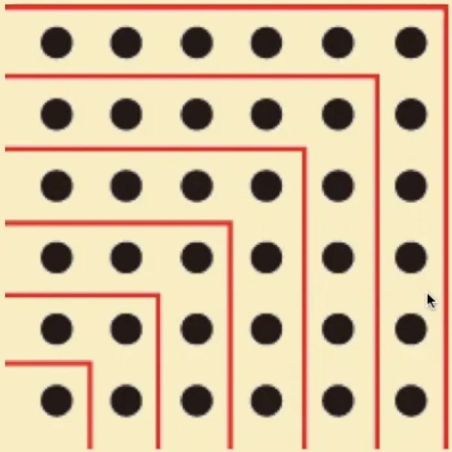

$$\Longrightarrow 1 + 3 + \cdots + (2n - 1) = n^2$$

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$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 16$$

$$\dots$$

$$1 + 2 + \cdots + (n-1) + n + (n-1) \cdots + 2 + 1 = n^2$$

Patterns in numbers

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$$3 + 5 = 8$$

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$$1 = 1$$

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$$7 + 9 + 11 = 27$$

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"There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that ... they grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behavior, and that they obey these laws with almost military precision."

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Today: We will explore the patterns between prime numbers and a sequence of numbers

$$\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \ldots$$

The numbers $\theta_1, \theta_2, \theta_3 \ldots$

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$$\theta_3 = 25.010857\ldots.$$

$$\theta_4 = 30.424876\ldots.$$

$$\theta_5 = 32.935061\ldots$$

Plot of $f(t)$

Let

$$f(t) = -\left(\cos(\theta_1 \log t) + \cdots + \cos(\theta_{10} \log t)\right)$$

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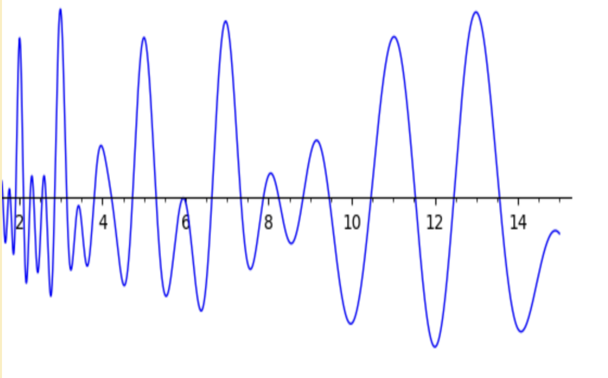

Plot of $f(t)$

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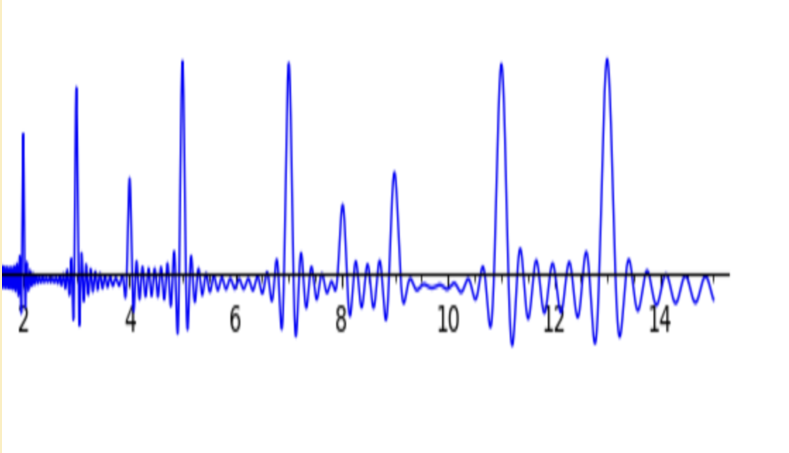

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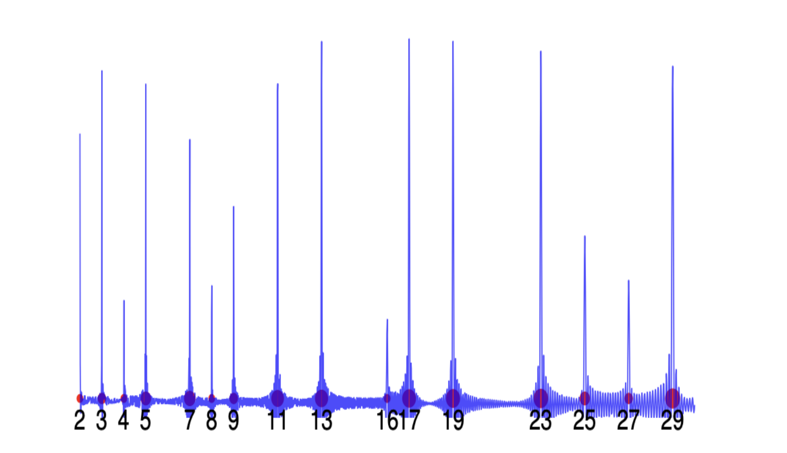

Plot of $g(t)$

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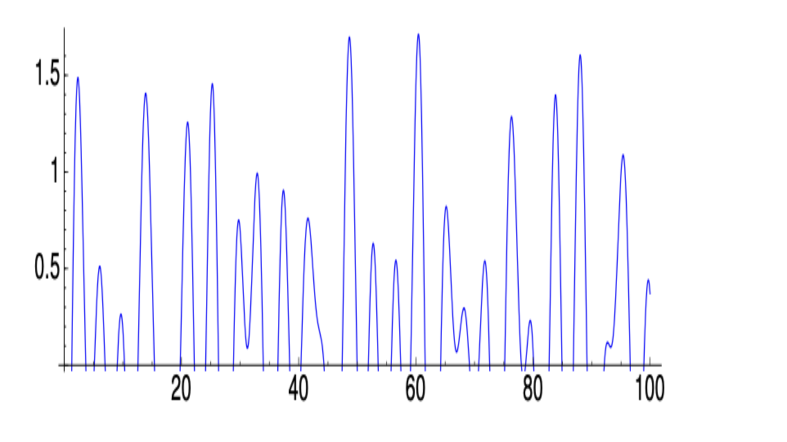

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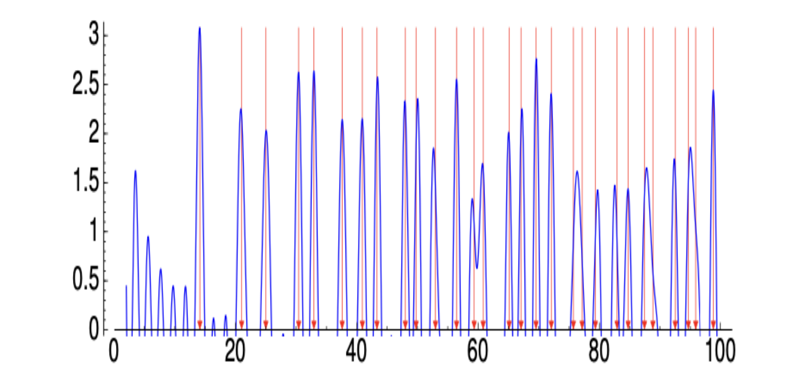
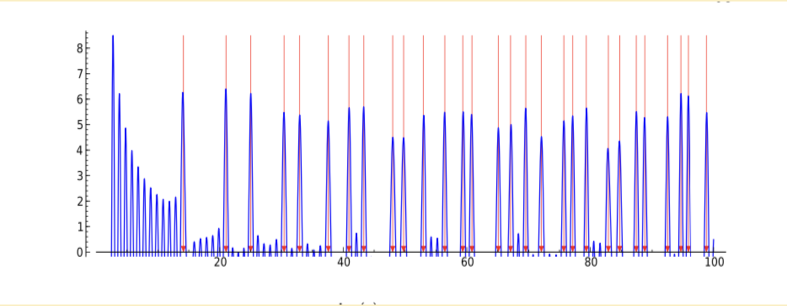


If $C = 500$, $g(t)$ looks like



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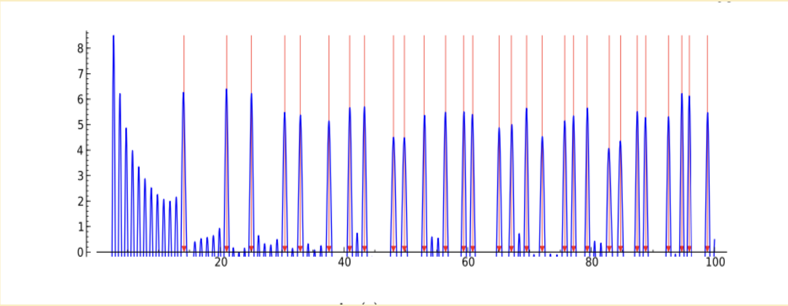

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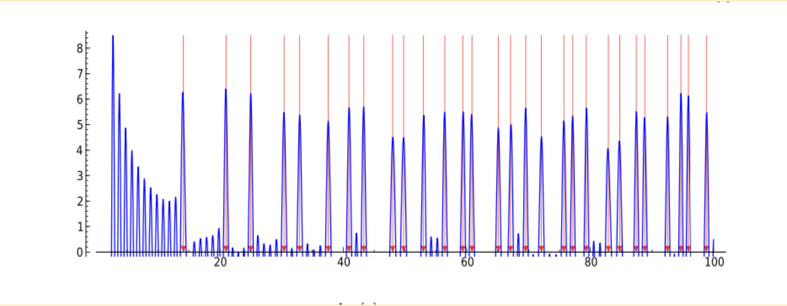

The red spikes occur at 14.134725, 21.022039, 25.010857 . . . . These are precisely the numbers $\theta_1, \theta_2, \theta_3 \ldots$!

Primes and $\theta_1, \theta_2, \ldots$

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What is the connection to prime numbers?

Riemann gave a profound answer to these questions in 1859: both primes and the numbers $\theta_1, \theta_2, \ldots$ are realted to a third object: the zeta function $\zeta(s)$.

The Riemann zeta function

Definition

For a complex number s, we define the Riemann zeta function by the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

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This series converges when $\mathrm{Re}(s) > 1$, and defines a holomorphic function (analogue of a differentiable function) on this region.

The Euler product

Theorem (Euler)

We have that

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$
$$= \left(1 - \frac{1}{2^s}\right)^{-1} \cdot \left(1 - \frac{1}{3^s}\right)^{-1} \cdot \left(1 - \frac{1}{5^s}\right)^{-1} \cdots$$

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This expression takes the shape

$$\boxed{\text{Sum over natural numbers}} = \boxed{\text{Product over primes}}$$

Values of ζ

Euler also computed

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}.$$

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In general, he showed that

$$\zeta(2k) = \frac{1}{1^{2k}} + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \cdots = \pi^{2k} \cdot x$$

for some $x \in \mathbb{Q}$.

Zeta at negative integers

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$$1^3 + 2^3 + 3^3 + 4^3 + \cdots = \frac{1}{120}$$

REMARQUES

SUR UN BEAU RAPPORT ENTRE LES SÉRIES DES PUISSANCES TANT DIRECTES QUE RÉCIPROQUES.

PAR M. L. EULER *).

I.

Le rapport, que je me propoſe de développer ici, regarde les ſommes de ces deux ſéries infinies générales:

$$\odot \;.\; 1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \&c.$$

$$☽ \;.\; \frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \frac{1}{7^n} - \frac{1}{8^n} + \&c.$$

Euler's functional equation

For $n = 1 + m$, Euler proved the relation

$$\frac{\odot}{☽} = -\frac{1 \cdot 2 \cdot 3 \cdots (n-1)}{(2^{n-1}-1)\pi^n}(2^n - 1)\cos\left(\frac{n\pi}{2}\right).$$

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- He also showed that we have a symmetry:

$$\zeta(s) \longleftrightarrow \zeta(1-s).$$

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More precisely,

$$\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

where $\Gamma(s)$ is the celebrated Gamma function defined by $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$.

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The imaginary parts of the zeros are exactly the θ_i's!

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$$\sum_{p^m \leq x} \log p = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{m=1}^{\infty} \frac{x^{-2m}}{2m} + \log(2\pi),$$

where the sum on the RHS is over all zeros ρ *of* $\zeta(s)$ *in the critical strip.*

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This formula is of the shape

$$\boxed{\textbf{Primes}} = \boxed{\textbf{Zeros}}$$

The Riemann Hypothesis

Let s be a zero of $\zeta(s)$ in the region $0 \leq \mathrm{Re}(s) \leq 1$. Then

$$\mathrm{Re}(s) = \frac{1}{2}.$$

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Let

$$Li(x) = \int_2^x \frac{dt}{\log t}.$$

Then we have that

$$\lim_{x\to\infty} \frac{\pi(x)}{Li(x)} = 1.$$

How good is this approximation?

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The Riemann Hypothesis implies that

$$\mathrm{Error}(x) \leq \frac{1}{8\pi}\sqrt{x}\log x \text{ for all } x \geq 2687.$$

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- At $x = 26863$, Team 3 catches up.

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- At $x = 26863$, Team 3 catches up.
- At $x = 26879$, Team 3 gets ahead.

Prime number races

We can divide all odd prime numbers into two teams: “Team 1” consisting of those prime numbers which are 1 mod 4, and ”Team 3” consisting of those prime numbers which are 3 mod 4.

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- For $267879 < x < 616841$, Team 3 is always in the lead.

Chebyshev's bias

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Shin-ya Koyama will be speaking at the Warwick Number Theory

Seminar on 24th Feb (3pm to 4pm in B3.02)!

Evidence for the Riemann Hypothesis

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that have

- Euler product
- Analytic continuation
- Functional Equation

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- This is an active area of research today and is known as the Langlands program.
- It is conjectured that every L-function comes from a class of objects known as **cuspdial automorphic representations**.
- The study of these objects has been fruitful so far and may lead to further progress...

Thank you!